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Advance Engineering Mathematics
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1. Using 3-digit floating-point arithmetic, apply the classical Gram-Schmidt algorithm to the set

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 10^{-3} \\ 10^{-3} \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 1 \\ 10^{-3} \\ 0 \end{pmatrix}, \quad \mathbf{x}_3 = \begin{pmatrix} 1 \\ 0 \\ 10^{-3} \end{pmatrix}.$$

show that the resulted vectors are not orthogonal. Explain why? Use the modified Gram-Schmidt algorithm to obtain a set of orthogonal vectors.

Modified Gram–Schmidt Algorithm

For a linearly independent set $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\} \subset \mathcal{C}^{m\times 1}$, the Gram–Schmidt sequence can be alternately described as

$$\mathbf{u}_k = \frac{\mathbf{E}_k \cdots \mathbf{E}_2\mathbf{E}_1\mathbf{x}_k}{\|\mathbf{E}_k \cdots \mathbf{E}_2\mathbf{E}_1\mathbf{x}_k\|} \text{ with } \mathbf{E}_1 = \mathbf{I}, \ \mathbf{E}_i = \mathbf{I} - \mathbf{u}_{i-1}\mathbf{u}_{i-1}^* \text{ for } i > 1,$$

and this sequence is generated by the following algorithm.

For $k = 1$: $\quad \mathbf{u}_1 \leftarrow \mathbf{x}_1 / \|\mathbf{x}_1\|$ and $\mathbf{u}_j \leftarrow \mathbf{x}_j$ for $j = 2, 3, \ldots, n$

For $k > 1$: $\quad \mathbf{u}_j \leftarrow \mathbf{E}_k\mathbf{u}_j = \mathbf{u}_j - \left(\mathbf{u}_{k-1}^*\mathbf{u}_j\right)\mathbf{u}_{k-1}$ for $j = k, k+1, \ldots, n$

$\qquad\qquad \mathbf{u}_k \leftarrow \mathbf{u}_k / \|\mathbf{u}_k\|$

2. Show that the sets

$$\left\{\frac{1}{\sqrt{\pi}}\right\} \cup \left\{\sqrt{\frac{2}{\pi}}\cos nt \mid n \in \mathbb{N}\right\}$$

and

$$\left\{\sqrt{\frac{2}{\pi}}\sin nt \mid n \in \mathbb{N}\right\}$$

are both complete orthonormal sets in $L^2(0,\pi)$.

3. (Hermite Polynomials) Consider the Hilbert space

$$H = \left\{f : \mathbb{R} \to \mathbb{R} \mid \int_{-\infty}^{\infty} e^{-x^2}|f(x)|^2\, dx < \infty\right\}$$

with the inner-product

$$(f,g) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-x^2} f(x)g(x)\, dx.$$

(a) Show that $f_n(x) = x^n$ belongs to H for every $n \in \{0\} \cup \mathbb{N}$.
(b) Apply the Gram-Schmidt process to the linearly independent set $\{f_n\}$ to obtain an orthonormal set h_n. Define

$$H_n(x) = \sqrt{2^n n!}\, h_n(x).$$

These are the *Hermite polynomials*. Compute H_0 and H_1.
(c) Prove Rodrigues' Formula:

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}.$$

4. Let $f, g \in L^2(-\pi, \pi)$ and let their Fourier series be given by

$$\begin{aligned} f &\sim \tfrac{a_0}{2} + \textstyle\sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt) \\ g &\sim \tfrac{c_0}{2} + \textstyle\sum_{n=1}^{\infty}(c_n \cos nt + d_n \sin nt). \end{aligned}$$

Show that

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)g(t)\, dt = \frac{a_0 c_0}{2} + \sum_{n=1}^{\infty}(a_n c_n + b_n d_n).$$

5. Compute the Fourier series of the function:

$$f(t) = \begin{cases} -1 & -\pi \leq t < 0 \\ 1 & 0 < t \leq \pi. \end{cases}$$

6. Compute the Fourier cosine series of the function $f(t) = \sin t$ on $[0, \pi]$.

7. (a) Compute the Fourier sine series and the Fourier cosine series of the function $f(t) = t$ on $[0, \pi]$.
(b) Evaluate:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^4}.$$

8. Compute the Fourier series of the function $f(t) = t$ on $[-\pi, \pi]$ and use it to compute the Fourier series for the function $f(t) = t^2$. Deduce that

$$1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots \;=\; \frac{\pi^2}{12}.$$

9. (a) Show that the function $f(t) = -\log|2\sin\frac{t}{2}|$ is in $L^1(-\pi, \pi)$.
(b) For $t \neq 2k\pi,\;\; k \in \mathbb{Z}$, show that

$$-\log|2\sin\frac{t}{2}| \;=\; \sum_{n=1}^{\infty} \frac{\cos nt}{n}.$$

(c) Deduce that

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \;=\; \log 2.$$

(d) Show that

$$-\log|2\cos\frac{t}{2}| \;=\; \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\cos nt}{n}.$$

(e) For $0 < t < \pi$, show that

$$\sum_{k=0}^{\infty} \frac{\cos(2k+1)t}{2k+1} \;=\; -\frac{1}{2} \log\tan\frac{t}{2}.$$